Line and Contour Integrals

Parametrization, Orientation, and a Fundamental Example

Lecture Notes

November 3, 2026

1 Line (Contour) Integrals

Let γ be a piecewise smooth curve, f continuous on $\gamma([a,b])$, with parametrization $z(t) = (x(t), y(t))$, $t \in [a,b]$.

Definition (line integrals)

$$\oint_\gamma f(z)\,dx := \int_a^b f(z(t))\,x'(t)\,dt, \qquad \oint_\gamma f(z)\,dy := \int_a^b f(z(t))\,y'(t)\,dt,$$

$$\int_\gamma f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt,$$

$$\oint_\gamma f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt = \oint_\gamma f(z)\,dx + i\oint_\gamma f(z)\,dy.$$

Properties

(1) **Independence of parametrization.** If $s : [c,d] \to [a,b]$ is increasing, piecewise differentiable, with $s(c) = a$, $s(d) = b$, then

$$\oint_\gamma f(z)\,dz = \int_c^d f(z(s(t)))\,z'(s(t))\,s'(t)\,dt.$$

(2) **Change of orientation.** If instead $s : [c,d] \to [a,b]$ is *decreasing*, piecewise differentiable, with $s(c) = b$, $s(d) = a$, then

$$\oint_\gamma f(z)\,dz = -\int_c^d f(z(s(t)))\,z'(s(t))\,s'(t)\,dt.$$

Notation: $-\gamma$ (or γ^-) denotes γ traversed in the opposite direction; then $\oint_{-\gamma} f\,dz = -\oint_\gamma f\,dz$.

(3) **Additivity.** If $\gamma_1 : [a,b] \to \mathbb{C}$, $\gamma_2 : [b,c] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(b)$, define $\gamma_1 + \gamma_2$ on $[a,c]$ by concatenation. Then

$$\oint_{\gamma_1+\gamma_2} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz.$$

(4) **Linearity.** For $\alpha, \beta \in \mathbb{C}$,

$$\oint_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\oint_\gamma f(z)\,dz + \beta\oint_\gamma g(z)\,dz.$$

Proofs of the properties

(1) Independence of parametrization. Let $\widetilde{\gamma}(t) = z(s(t))$ for $c \le t \le d$. On each smooth piece, the chain rule gives

$$\frac{d}{dt} z(s(t)) = z'(s(t))s'(t).$$

Using the definition of the contour integral and then the substitution $u = s(t)$, with $s(c) = a$ and $s(d) = b$, we obtain

$$\begin{aligned}\oint_{\widetilde{\gamma}} f(z)\,dz &= \int_c^d f(z(s(t)))\frac{d}{dt} z(s(t))\,dt \\ &= \int_c^d f(z(s(t)))z'(s(t))s'(t)\,dt \\ &= \int_a^b f(z(u))z'(u)\,du = \oint_\gamma f(z)\,dz.\end{aligned}$$

For piecewise smooth parametrizations, apply the calculation on the pieces and add. Thus an increasing change of parameter leaves the integral unchanged.

(2) Change of orientation. Again let $\widetilde{\gamma}(t) = z(s(t))$, now with s decreasing, $s(c) = b$ and $s(d) = a$. The chain rule is the same, but the substitution reverses the integration limits:

$$\begin{aligned}\oint_{\widetilde{\gamma}} f(z)\,dz &= \int_c^d f(z(s(t)))z'(s(t))s'(t)\,dt \\ &= \int_b^a f(z(u))z'(u)\,du \\ &= -\int_a^b f(z(u))z'(u)\,du = -\oint_\gamma f(z)\,dz.\end{aligned}$$

In particular, the reversed curve may be parametrized by $(-\gamma)(t) := \gamma(a + b - t)$, $a \le t \le b$. Taking $s(t) = a + b - t$ proves $\oint_{-\gamma} f(z)\,dz = -\oint_\gamma f(z)\,dz$.

(3) Additivity. The concatenated path is, explicitly,

$$(\gamma_1 + \gamma_2)(t) = \begin{cases}\gamma_1(t), & a \le t \le b, \\ \gamma_2(t), & b \le t \le c.\end{cases}$$

It is continuous at b because $\gamma_1(b) = \gamma_2(b)$. Splitting the parameter integral at b gives

$$\begin{aligned}\oint_{\gamma_1+\gamma_2} f(z)\,dz &= \int_a^b f(\gamma_1(t))\gamma_1'(t)\,dt + \int_b^c f(\gamma_2(t))\gamma_2'(t)\,dt \\ &= \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz.\end{aligned}$$

(4) Linearity. By the definition and the linearity of integration of complex-valued functions of a real variable,

$$\begin{aligned}\oint_\gamma (\alpha f(z) + \beta g(z))\,dz &= \int_a^b [\alpha f(z(t)) + \beta g(z(t))]z'(t)\,dt \\ &= \alpha\int_a^b f(z(t))z'(t)\,dt + \beta\int_a^b g(z(t))z'(t)\,dt \\ &= \alpha\oint_\gamma f(z)\,dz + \beta\oint_\gamma g(z)\,dz.\end{aligned}$$

□

2 A Very Important Example

Let $\gamma : [0, 2\pi] \to \mathbb{C}$, $z(t) = a + re^{it}$ (the circle of radius r about a, counterclockwise).

Key computation

For $n \in \mathbb{Z}$,

$$\oint_\gamma (z-a)^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1. \end{cases}$$

Proof.

$$\oint_\gamma (z-a)^n \, dz = \int_0^{2\pi} r^n e^{int} \cdot ire^{it} \, dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = ir^{n+1} \int_0^{2\pi} \Big[\cos\big((n+1)t\big) + i \sin\big((n+1)t\big) \Big] dt.$$

If $n \neq -1$, this evaluates to

$$ir^{n+1} \cdot \frac{1}{n+1} \Big[\sin\big((n+1)t\big) - i\cos\big((n+1)t\big) \Big]_0^{2\pi} = 0,$$

since the bracket is 2π-periodic. If $n = -1$, the integrand is identically i, so the integral is $i \cdot 2\pi = 2\pi i$. □

Heuristic explanation. For $n \neq -1$, $(z-a)^n = \left(\dfrac{(z-a)^{n+1}}{n+1} \right)'$ is the derivative of a single-valued function, so its integral around any closed loop vanishes – a complex analogue of the Fundamental Theorem of Calculus. For $n = -1$, formally $(z-a)^{-1} = \big(\log(z-a)\big)'$, but log is multivalued: going once around γ, its imaginary part (the argument) increases by 2π, so

$$\oint_\gamma (z-a)^{-1} \, dz \text{ “=” } \big[\log(z-a)\big]_{\text{after one loop}} - \big[\log(z-a)\big]_{\text{before}} = 2\pi i - 0 = 2\pi i.$$